Oct. 27th 2014

Celebrating Tony's 60th birthday Anniversary

Tony, happy 60th birthday!

Unified Wave Model for Progressive Waves in Finite Water Depth

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Outline

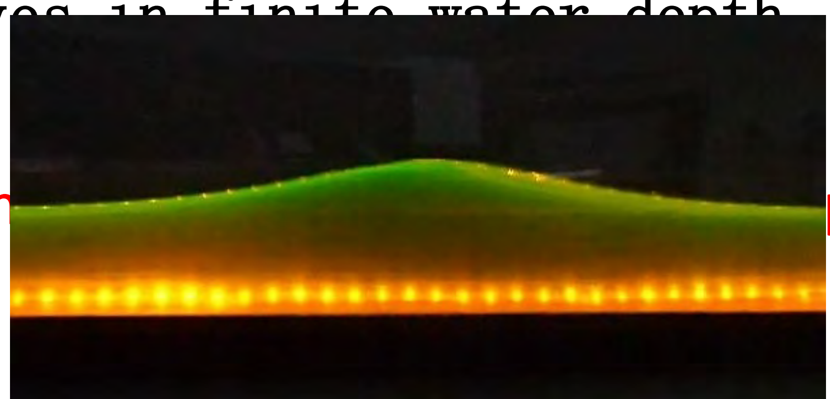
1. **Motivations**
2. **Unified Wave Model (UWM)**
3. **UWM: Smooth Waves**
4. **UWM: Peaked Waves**
5. **Conclusions and discussions**

1. Motivations

Models for smooth progressive water waves

- 1845: solitary wave (J. S. Russell)
- 1872: Boussinesq equation (J. Boussinesq)
- 1894: limiting progressive waves (G.G. Stokes)
- 1895: KdV equation (D. J. Korteweg and G. de Vries)
- 1970s: waves in finite water depth (J.D. Fenton)

Good agreemen



periments

1. Motivations

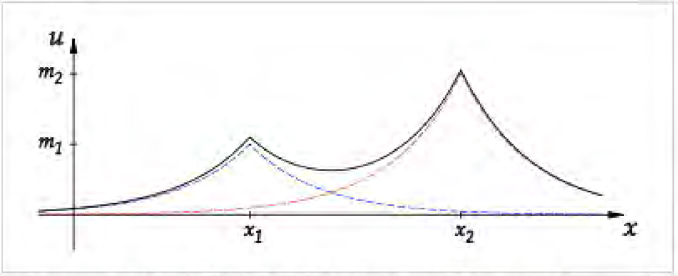
- **Cammasa-Holm equation (PRL,1993) :**

$$u_t + 2\omega u_X - u_{XXt} + 3uu_X = 2u_X u_{XX} + uu_{XXX},$$

with a peaked solitary wave:

$$u(X,t) = c\ \exp(-|X - c\,t|),$$

where ω is a constant related to the critical phase speed of shallow water waves



Two-peakon wave profile (solid curve) formed by adding two peakons (dashed curves): $u = m_1\, e^{-|x-x_1|} + m_2\, e^{-|x-x_2|}$

1. Motivations

- **Mathematically, the CH equation is integrable and bi-Hamiltonian, thus possesses an infinite number of conservation laws in involution (Camassa-Holm,1993);**
- **Physically, unlike the KdV equation and Boussinesq equation, the CH equation can model phenomena of soliton interaction and wave breaking (Constantin, 2000)**
- **A few researchers even believed that "it has the potential to become the new master equation for shallow water wave theory" (Fuchssteiner, 1996)**

1. Motivations

Some open questions

- **How about finite water depth?**
- **How about exact wave equations?**
- **Are the peaked solitary waves consistent with the smooth waves in theory?**
- **Why can we not observe them in experiments?**
- **Can we gain more information so as to observe the peaked/cusped waves in experiments?**

2. Unified Wave Model (UWM)

- **Smooth waves:**

  infinitely differentiable everywhere

Reason: solutions of Laplace equation are infinitely differentiable everywhere.

- **Peaked solitary waves:** $u(X,t) = c\ \exp(-|X - c\,t|),$

  non-smooth at crest !

How to handle such kind of non-smoothness?

2. Unified Wave Model (UWM)

Consider a progressive surface gravity wave propagating on a horizontal bottom with a constant phase speed *c* and a permanent form in a finite water depth *D*. We solve the problem in the frame moving with the phase speed *c*.

- **Assume that the wave elevation has a symmetry about the crest at $x = 0$;**
- **Assume that, in the domain $x > 0$ and $x < 0$, the fluid is inviscid and incompressible, the flow is irrotational, and surface tension is neglected;**
- **However, the flow at $x = 0$ is not absolutely necessary to be irrotational.**

2. Unified Wave Model (UWM)

(1) Symmetry:

$$\zeta(x) = \zeta(-x), \quad u(x,z) = u(-x,z), \quad v(x,z) = -v(-x,z),$$

which leads to the boundary conditions

(a) $\zeta(x), u(x,z)$ are continuous at $x = 0$

(b) $v(0,z) = 0$, since $v(0,z) = -v(0,z)$

for both of the smooth and peaked waves.

So, we need governing equations and free surface conditions only in the domain $0 < x < +\infty$

2. Unified Wave Model (UWM)

(2) Equations in the domain $0 < x < +\infty$

$$\nabla^2\phi(x,z) = 0, \qquad z \le \zeta(x), 0 < x < +\infty,$$

with free surface boundary conditions

$$\alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} - \alpha \frac{\partial}{\partial x}(\nabla\phi\cdot\nabla\phi) + \nabla\phi\cdot\nabla\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi\right) = 0, \qquad 0 < x < +\infty,$$

$$\zeta - \alpha\frac{\partial \phi}{\partial x} + \frac{1}{2}\nabla\phi\cdot\nabla\phi = 0, \qquad 0 < x < +\infty,$$

bottom condition: $\frac{\partial \phi}{\partial z} = 0, \qquad z = -1,\ 0 < x < +\infty,$

open boundary condition: $u(0,z) = \lim_{x\to 0}\frac{\partial \phi}{\partial x} = U(z), \qquad z \le \zeta(x),$

periodic waves: $\phi(x,z) = \phi(x+\lambda, z),$

solitary waves: $\phi(\pm\infty, z) = 0,$

2. Unified Wave Model (UWM)

Mathematically, Laplace equation (with bottom condition) has two kinds of solutions:

(1) traditional base function

$$\cosh[nk(z+1)]\ \sin(nkx), \quad n \geq 1,$$

corresponding to smooth waves

(2) evanescent base function (Massel, 1983)

$$\cos[nk(z+1)]\ \exp(-nkx), \ \ n \geq 1,\ k > 0,\ 0 \leq x < +\infty,$$

corresponding to peaked solitary waves

3. UWM: smooth waves

The smooth potential function

$$\phi(x,z)=\sum_{n=1}^{+\infty} b_n \cosh\left[nk(z+1\right]\sin(nkx)$$

automatically satisfy all symmetry conditions

$$\zeta(x)=\zeta(-x),\quad u(x,z)=u(-x,z),\quad v(x,z)=-v(-x,z),$$

Therefore, all traditional smooth propagating waves can be derived in the frame of the Unified Wave Model (UWM)

It also supports the correctness of the UWM in mathematics.

4. UWM: peaked waves

The velocity potential of peaked waves

$$\phi(x,z) = \sum_{n=1}^{+\infty} a_n \cos[nk(z+1)] \exp(-nkx), \qquad x > 0$$

does not automatically satisfy the symmetry

$$\zeta(x) = \zeta(-x), \quad u(x,z) = u(-x,z), \quad v(x,z) = -v(-x,z),$$

and the restrict condition

$$v(0,z) = 0.$$

Therefore, the restricted condition $v(0,z) = 0.$ **must be enforced to be satisfied**

4. UWM: peaked waves

Linear theory of peaked waves:

$$\nabla^2\phi(x,z) = 0, \qquad z \le \zeta(x), 0 < x < +\infty,$$

$$\alpha^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial z} = 0, \qquad \text{on } z = 0,\ 0 < x < +\infty,$$

$$\zeta(x) = \alpha\frac{\partial\phi}{\partial x}\Big|_{z=0}, \quad 0 < x < +\infty,$$

solution:

$$\phi^+(x,z) = \alpha A\cos[k(z+1)]\, e^{-kx}, \quad 0 < x < +\infty,$$

phase speed: $$\alpha^2 = \frac{\tan k}{k}, \qquad n\pi < k < n\pi + \frac{\pi}{2},$$

Given $\alpha = \frac{c}{\sqrt{gD}}$, there exist an infinite number of solutions

$$\mathbf{K}_\alpha = \left\{k_n : \alpha^2 = \frac{\tan k_n}{k_n}, n\pi < k_n < n\pi + \frac{\pi}{2}, n = 0, 1, 2, 3, \cdots\right\}$$

4. UWM: peaked waves

Linear theory of peaked waves

Surface elevation: $$\zeta(x) = H_w \, e^{-k_\nu |x|}, \qquad -\infty < x < +\infty.$$

Camassa-Holm's peaked wave is only a special case of our peaked solitary waves !

This indicates that the UWM is reasonable

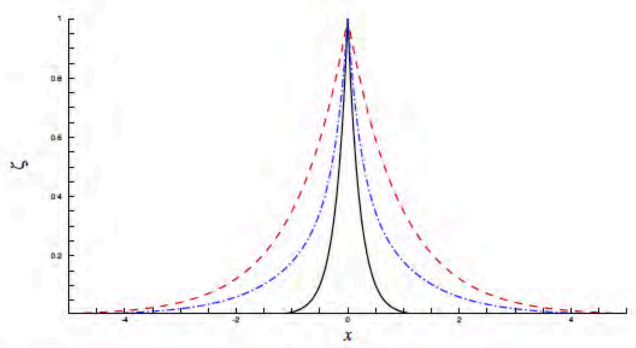
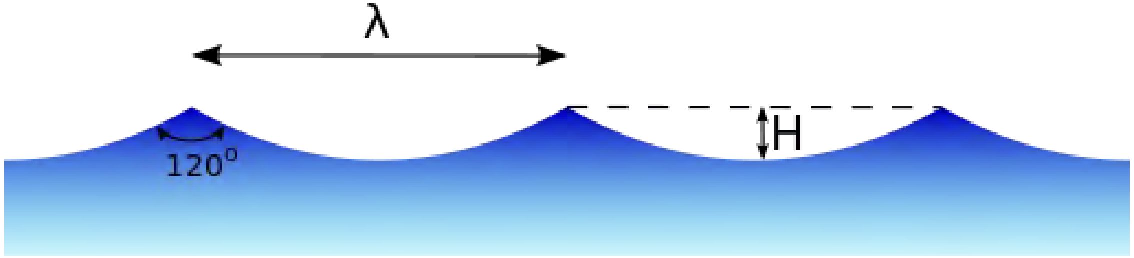


Figure 1: $\zeta(x)/H_w$ in the case of $k_0 = \pi/3, k_1 = 4.58117$ with $\alpha = 3^{3/4}/\sqrt{\pi}$. Dashed line: $\exp(-k_0|x|)$; Solid line: $\exp(-k_1|x|)$; Dash-dotted line: $[\exp(-k_0|x|) + \exp(-k_1|x|)]/2$.

4. UWM: peaked waves

G.G. Stokes (1894):

The smooth propagating periodic waves tend to have a corner crest at the limiting wave amplitude, and thus become non-smooth.



Thus, non-smooth waves are acceptable in the frame of inviscid fluid.

4. UWM: peaked waves

Linear theory of peaked waves

Velcoity:

$$\frac{u}{U_0} = \frac{\cos[k_\nu(z+1)]e^{-k_\nu|x|}}{\cos(k_\nu)}, \qquad x \in (-\infty, +\infty),$$

$$v^+(x,z) = \frac{\partial \phi^+}{\partial z} = \frac{\alpha k_\nu H_w \sin[k_\nu(z+1)]e^{-k_\nu x}}{\sin k_\nu}$$

$$v^-(x,z) = -v^+(-x,z) = -\frac{\alpha k_\nu H_w \sin[k_\nu(z+1)]e^{k_\nu x}}{\sin k_\nu}, \quad -\infty < x < 0$$

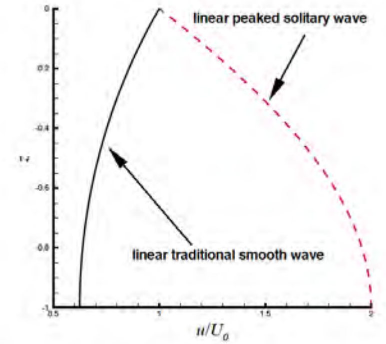


Figure 2: Velocity profile u/U_0 at $x = 0$ in the case of $k_0 = \pi/3$ with $U_0 = H_w/\alpha$. Solid line: periodic Airy wave; Dashed line: linear peaked solitary wave.

(1) $u(x,z)$ increases from surface to bottom, and is continuous at $x = 0$

(2) $v(x,z)$ decreases from surface to bottom, and is discontinuous at $x = 0$, say, $\lim_{x\to 0} v^+ = -\lim_{x\to 0} v^- \neq 0.$

Thus, the condition $v(0,z) = 0$ is enforced.

4. UWM: peaked waves

Lamb (Hydrodynamics, page 371):

"the tangential velocity changes sign as we cross the surface", but "in reality the discontinuity, if it could ever be originated, would be immediately abolished by viscosity"

Thus, the discontinuity of velocity is also acceptable in the frame of inviscid fluid

Kinetic energy distribution

Smooth periodic waves

$$E_k = \frac{1}{2}\left(u^2 + v^2\right) = \left[\frac{\alpha H_w k}{4\sinh(k)}\right]^2 \{\cosh[2k(1+z)] + \cos(2kx)\}$$

- Decays **exponentially** from surface to bottom
- varies **periodically** in the *x* direction

Peaked waves

$$E_{k_\nu} = \frac{1}{2}\left(u^2 + v^2\right) = \frac{(\alpha H_w k_\nu)^2}{2\sin^2(k_\nu)} \exp(-2k_\nu |x|)$$

- keeps **constant** from surface to bottom
- Decays **exponentially** in the *x* direction

4. UWM: peaked waves

Nonlinear theory of peaked waves

$$\nabla^2\phi(x,z)=0, \qquad z\le\zeta(x), 0<x<+\infty,$$

with fully nonlinear surface conditions

$$\alpha^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial z}-\alpha\frac{\partial}{\partial x}(\nabla\phi\cdot\nabla\phi)+\nabla\phi\cdot\nabla\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi\right)=0, \qquad 0<x<+\infty,$$

$$\zeta-\alpha\frac{\partial\phi}{\partial x}+\frac{1}{2}\nabla\phi\cdot\nabla\phi=0, \qquad 0<x<+\infty,$$

the bottom condition

$$\frac{\partial\phi}{\partial z}=0, \qquad z=-1,\ 0<x<+\infty,$$

and other conditions

$$u(0,z)=\lim_{x\to 0}\frac{\partial\phi}{\partial x}=U(z), \qquad z\le\zeta(x),$$

$$v(0,z)=0. \qquad \phi(\pm\infty,z)=0,$$

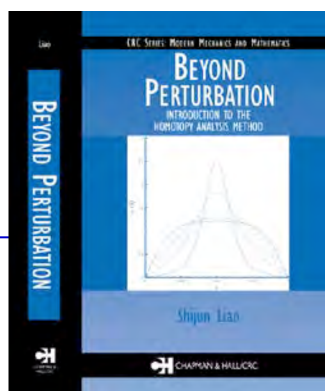
4. UWM: peaked waves

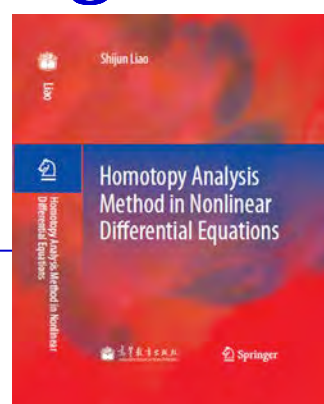
Nonlinear theory of peaked waves

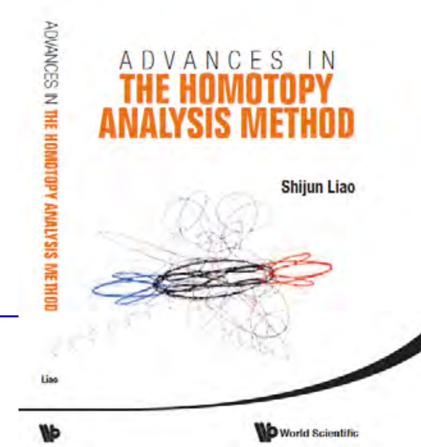
These nonlinear PDEs are solved by means of homotopy analysis method (HAM)

Advantages of the HAM:

- **Independent of physical small parameters**
- **Guarantee of convergence**







Applications of the HAM

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Nonlinear progressive waves in water of finite depth — an analytic approximation

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On the steady-state fully resonant progressive waves in water of finite depth

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4. UWM: peaked waves

Convergent series solutions:

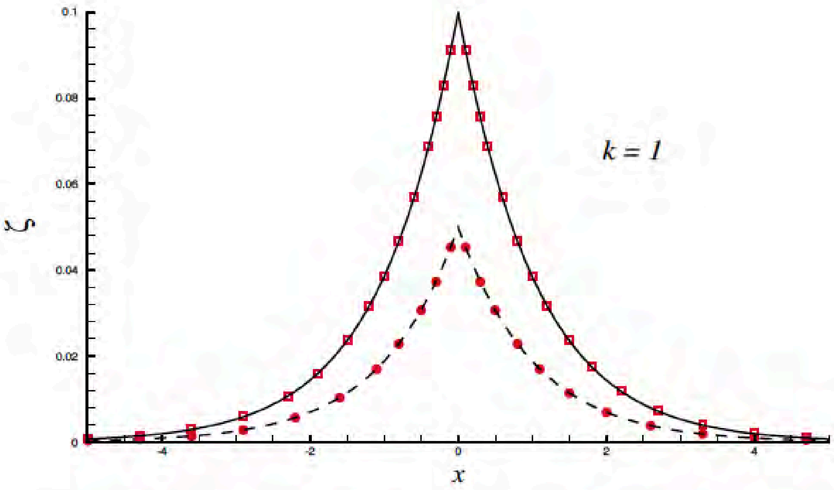

Figure 8: Analytic approximations of elevation of the peaked solitary waves when $k = 1$ (corresponding to $c/\sqrt{gD} = 1.24796$). Solid line: 5th-order approximation when $H_w = 0.1$ given by $c_\phi = -0.5$ and $c_\eta = -1$; Filled circles: 25th-order approximation when $H_w = 0.1$ given by $c_\phi = -0.5$ and $c_\eta = -1$; Dashed line: 5th-order approximation when $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$; Open circles: 25th-order approximation when $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$.

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Do peaked solitary water waves indeed exist?

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A B S T R A C T

Many models of shallow water waves, such as the famous Camassa–Holm equation, admit peaked solitary waves. However, it is an open question whether or not the widely accepted peaked solitary waves can be derived from the fully nonlinear wave equations. In this paper, a unified wave model (UWM) based on the symmetry and the fully nonlinear wave equations is put forward for progressive waves with permanent form in finite water depth. Different from traditional wave models, the flows described by the UWM are not necessarily irrotational at crest, so that it is more general. The unified wave model admits not only the traditional progressive waves with smooth crest, but also a new kind of solitary waves with peaked crest that include the famous peaked solitary waves given by the Camassa–Holm equation. Besides, it is proved that Kelvin's theorem still holds everywhere for the

4. UWM: peaked waves

Nonlinear theory of peaked waves

- **The phase speed of peaked waves has nothing to do with wave height, say, the peaked waves are non-dispersive!**
- **The kinetic energy is almost the same from surface to bottom**
- **There exists the velocity discontinuity at crest**

Comparison of smooth and peaked waves

Smooth waves

- Smooth everywhere
- Dispersive
- Kinetic energy decays exponentially from surface to bottom

Peaked waves

- Non-smooth at crest
- Non-dispersive
- Kinetic energy is almost the same from surface to bottom

A new theoretical explanation to rogue wave

The rogue wave can suddenly appear on ocean even when "the weather was good, with clear skies and glassy swells", as reported by Graham (2000) and Kharif (2003).

A new explanation:

Peaked solitary waves with small wave height and different phase speed may suddenly create a rogue wave somewhere, since they are non-dispersive

Relation between peaked and cusped waves

Using the non-dispersive property of peaked waves, it is proved that a cusped solitary wave is consist of an infinite number of peaked solitary waves

$$\eta(x) = \frac{H_w}{\zeta(\beta)} \sum_{n=1}^{+\infty} \frac{1}{n^\beta} \exp(-k_{n-1}|x|),$$

$$\alpha^2 = \frac{\tan k_n}{k_n}, \quad n\pi \le k_n \le n\pi + \frac{\pi}{2}, \ \ n \ge 0,$$

$$1 < \beta \le 2$$

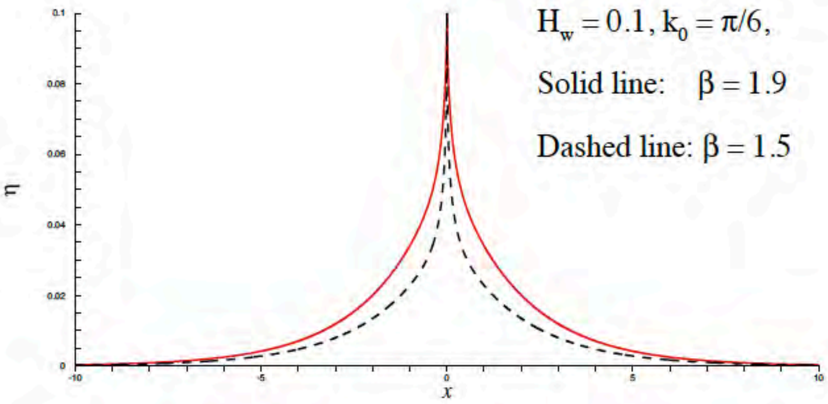

FIG. 1. Cusped solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/4}/\sqrt{\pi}$). Solid line: $\beta = 1.9$; Dashed line: $\beta = 1.5$.

5. Conclusions and discussions

(1) The UWM gives not only the traditional smooth progressive waves but also the famous Camassa-Holm's peaked waves

Therefore, the UAM is reasonable and more general from mathematical viewpoint

5. Conclusions and discussions

(2) The UWM admits not only smooth progressive waves but also peaked/cusped solitray waves in finite water depth

Therefore, the peaked/cusped solitray waves are consistent with smooth waves

5. Conclusions and discussions

(3) The peaked solitary waves have many unusual characteristics:

- **Phase speed is independent of wave height (non-dispersive)**
- **Kinetic energy is almost the same from surface to bottom**
- **Velocity discontinuity at crest**

These information are helpful for possible experimental observations of them in future.

5. Conclusions and discussions

(4) Using the non-dispersive property of peaked solitary waves,

(a) a simple but elegant relationship between peaked and cusped waves is given,

(b) a new theoretical explanation of rogue wave is suggested.

5. Conclusions and discussions

Models for progressive water waves

- **1845：solitary wave（J. S. Russell）**
- **1872：Boussinesq equation（J. Boussinesq）**
- **1894：limiting progressive waves （G.G. Stokes）**
- **1895：KdV equation （D. J. Korteweg and G. de Vries）**
- **1970s: waves in finite water depth （J.D. Fenton）**
- **1993: Camassa-Holm equation (Camassa and Holm)**
- **2014: Unified Wave Model (UWM)**

UWM can describe the smooth and non-smooth progressive waves of all previous models in shallow and finite depth of water!

Thank You!

Tony, happy 60’s Birthday Anniversary!